Courtesy of A. Gostomelskiy and D. V. Kozlov.

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THE VEDERNIKOV NUMBER

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ABSTRACT. The Vedernikov number of open-channel hydraulics is reviewed, explained, and clarified. With the Froude number, they constitute an inseparable pair, more so now that their relation with β, the exponent of the discharge-area rating (**V**/**F** = β - 1) has been clearly identified and its usefulness in channel design amply demonstrated. A calculator for β and **V** substantially increases the utility of the theory, making possible the avoidance of roll waves *at the design stage*.

1. INTRODUCTION

The Vedernikov number is one of *four* dimensionless numbers, or ratios, in open-channel flow. Two of these numbers, the Froude and Vedernikov numbers, are ratios of velocities; the other two, the Reynolds and dimensionless wavenumber of Ponce and Simons are ratios of diffusivities. These are the only dimensionless numbers that can be formulated with the *three* velocities and *three* diffusivities identified by **Ponce (1979)**. A recent article has helped throw additional light onto these fundamental concepts (**Ponce, 2023a**).

The Froude and Reynolds numbers have been well established for more than a century (Chow, 1959). Unfortunately, recognition of the Vedernikov number has lagged behind. This may be attributed to the fact that Chow chose to discuss the Vedernikov number *only once* in his authoritative textbook "*Open-Channel Hydraulics*", in Chapter 8, entitled "Theoretical Concepts of Boundary Layer, Surface Roughness, Velocity Distribution, and Instability of Uniform Flow." Experience shows that few pracficing hydraulic engineers may have actually read Chapter 8 *completely*, effectively consigning the Vedernikov number to the relative obscurity that it has endured over the past half century.

Ponce (1991a) has presented the Froude (**F**) and Vedernikov (**V**) numbers as essentially two parts of the same story, arguing convincingly for their parallel treatment, which Chow (1959) somehow missed. In open-channel hydraulics, the two numbers constitute a veritable duality, because their ratio **V**/**F** is ostensibly equal to (β - 1), in which β is the exponent of the discharge-flow area rating $Q = \alpha A^{\beta}$. The value of β is extremely important because it encapsulates not only the Froude and Vedernikov numbers, but also the channel's boundary friction and cross-sectional shape. These propositions will now be substantiated.

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2. THE VEDERNIKOV NUMBER

In order to appropriately describe the Vedernikov number, we must first define three velocities relevant to open-channel flow: (1) the mean velocity of the steady flow u, (2) the relative celerity of kinematic waves v, and (3) the relative celerity of dynamic waves w. Celerity is the velocity of a wave, as opposed to the mean velocity of the steady flow. Kinematic waves are the "long" waves of **Seddon (1900)**; dynamic waves are the "short" waves of **Lagrange (1788)**.

The Froude number is defined as **F** = u /w, the ratio of the mean velocity of the steady flow to the relative celerity of dynamic waves. The threshold value **F** = 1, referred to as critical flow, separates subcritical flow (**F** < 1) from supercritical flow **(F** > 1). At critical flow, the propagation of short surface waves switches from moving in an upstream direction (**F** < 1) to moving downstream (**F** > 1). The Froude number is normally used to describe steady flow; however, its definition (u/w) reveals that it also describes unsteady flow, albeit only the dynamic waves (**Ponce, 2023b**).

The Vedernikov number is defined as **F** = v /w, the ratio of the relative celerity of kinematic waves to the relative celerity of dynamic waves. The threshold value **V** = 1, referred to as *neutral* or *neutrally stable* flow, separates *stable* flow (**V** < 1) from *unstable* flow **(V** > 1). At neutral flow, kinematic and dynamic waves travel with the same celerity; for stable flow, dynamic waves travel *faster* than kinematic waves; for unstable flow, kinematic waves travel *faster* than dynamic waves.

The definition of the Vedernikov number, **V** = *v* /*w*, reflects the unmistakable competition between kinematic and dynamic waves (**Ponce, 2023b**). Unlike the Froude number, which considers only dynamic waves, the Vedernikov number compares the two types of waves and determines that the flow is either stable for **V** < 1, or unstable for **V** > 1. In hydraulic engineering practice, flow instability is a necessary, but not sufficient, condition for the occurrence of roll waves (**Ponce and Choque Guzman, 2019**) (Fig. 1).

Cornish (1907)

Fig. 1 Early fotograph of a train of roll waves in the Swiss Alps.

3. HISTORICAL BACKGROUND

The original development of the concept goes back to the work of Vedernikov, translated from the Russian language (**Vedernikov, 1945**; **1946**). At about the same time, Craya (1945) published a paper with a related content in the French journal *La Houille Blanche*. However, Craya's paper on the subject of flow instability was published in the English language only seven years later (**Craya, 1952**).

The name *Vedernikov number* originated with **Powell (1948)**, who stated: *"This criterion, which I am calling the Vedernikov number..."* Vedernikov's work, which unfortunately was not very straightforward in its original form, was clarified by **Craya (1952)**, who unmistakably stated that the criterion for flow instability is when the Seddon celerity *exceeds* the Lagrange celerity.

Chow (1959) sought to include the concept of Vedernikov number in his authoritative textbook. He described the concept in terms of the frictional and cross-sectional properties of the channel, essentially echoing Vedernikov's work on hydrodynamic instability (**Chow: Chapter 8, Section 8**).

Almost three decades later, the matter was substantially clarified by **Ponce (1991a)**, who simplified Vedernikov's original work by expressing the Vedernikov number, like the Froude number, solely in terms of the flow's mean velocity u and the relative wave celerities v and w. In fact: **F** = u / w and **V** = v / w. The remaining ratio, **V** / **F** = v /u, is identified as (β - 1), in which β is the exponent of the discharge-flow area rating $Q = \alpha A^{\beta}$ (**Box A**).

Box A. Relation of β, the exponent of the rating, to the Froude and Vedernikov numbers.

1. Discharge-flow area rating: $Q = \alpha A^{\beta}$.
2. Seddon celerity: dQ / dA (**Seddon, 1900**).
3. Seddon celerity (kinematic wave celerity): $dQ / dA = \alpha \beta A^{\beta - 1} = \beta (Q / A) = \beta u$.
4. Relative kinematic wave celerity: $v = \beta u - u = (\beta - 1) u$.
5. Ratio **V** / **F** = $(v / w) / (u / w) = v / u = \beta - 1$.
6. Therefore, β completes the triad of dimensionless parameters, encapsulating both the Froude and Vedernikov numbers.

Ponce and Simons (1977) have confirmed that **F** = 2 describes neutrally stable flow for the case of Chezy friction in hydraulically wide channels. In this case, **F** = 2 is equivalent to **V** = 1. Therefore, when taken together, the Froude and Vedernikov numbers describe the complete behavior of unsteady open-channel flow. Their distinctive ratio (β - 1), due to its conciseness and effectivess, makes β perhaps the most significant parameter in the entire field of unsteady open-channel flow.

4. EFFECT OF BOUNDARY FRICTION AND CROSS-SECTIONAL SHAPE

Given that **V**/**F** = (β - 1), a neutral stability Froude number may be defined as $\mathbf{F}_{ns}$ = 1 / (β - 1). For example, for Chezy friction in a hydraulically wide channel, for which β = 1.5, it follows that $\mathbf{F}_{ns}$ = 2, confirming the findings of **Ponce and Simons (1977)**. It is seen that since $\mathbf{F}_{ns}$ is a function solely of β, β encapsulates both Froude and Vedernikov numbers into *only one* parameter, the only one fully able to characterize hydrodynamic instability.

Table 1 shows values of β and $\mathbf{F}_{ns}$ for selected combinations of boundary friction and cross-sectional shape (**Ponce, 2014**). The value of β is seen to vary in the range $3 \geq \beta \geq 1$, for a high of β = 3 for laminar sheet flow, and a low of β = 1 for the *inherently stable* channel shape (**Ponce and Porras, 1995**). Accordingly, the value of $\mathbf{F}_{ns}$ varies in the range $0.5 \leq \mathbf{F}_{ns} \leq \infty$, for a low of $\mathbf{F}_{ns}$ = 0.5 for laminar sheet flow, and a high of $\mathbf{F}_{ns} = \infty$ for the inherently stable channel shape. Table 1 completes the description and assessment of hydrodynamic instability for any type of friction and cross-sectional shape.

TABLE 1. Values of neutral stability Froude number F_{ns} for selected values of β.

β	β - 1	Boundary friction	Cross-sectional shape	F_{ns}
3	2	Laminar	Hydraulically wide	1/2
8/3	5/3	Mixed laminar-turbulent (25% turbulent Manning)	Hydraulically wide	3/5
21/8	13/8	Mixed laminar-turbulent (25% turbulent Chezy)	Hydraulically wide	8/13
7/3	4/3	Mixed laminar-turbulent (50% turbulent Manning)	Hydraulically wide	3/4
9/4	5/4	Mixed laminar-turbulent (50% turbulent Chezy)	Hydraulically wide	4/5
2	1	Mixed laminar-turbulent (75% turbulent Manning)	Hydraulically wide	1
15/8	7/8	Mixed laminar-turbulent (75% turbulent Chezy)	Hydraulically wide	8/7
5/3	2/3	Turbulent Manning	Hydraulically wide	3/2
3/2	1/2	Turbulent Chezy	Hydraulically wide	2
4/3	1/3	Turbulent Manning	Triangular	3
5/4	1/4	Turbulent Chezy	Triangular	4
1	0	Any	Inherently stable	∞

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5. VEDERNIKOV NUMBER AND CHANNEL DIFFUSIVITY

Flood routing in open channels entails the calculation of two distinct physical processes: Convection and diffusion. Convection is a first-order process; diffusion is a second-order process. Convection is characterized by a convective velocity, or celerity; diffusion is characterized by the channel's hydraulic diffusivity. Since **Hayami's (1951)** seminal work, the expression for hydraulic diffusivity has seen a gradual change, as the subject matured and more information became available. The latest expression for hydraulic diffusivity, in terms of the Vedernikov number, is due to **Ponce (1991a)**. For the record, we retell the historical development in **Box B**.

Box B. Historical development of hydraulic diffusivity (**Nuccitelli and Ponce, 2014**).

1. The concept was originated by **Hayami (1951)**, who expressed the hydraulic diffusivity as $\nu_h = (u_o\, d_o) / (2S_o) = q_o / (2S_o)$, in which u_o = equilibrium flow velocity, d_o = equilibrium flow depth, q_o = equilibrium unit-width discharge, and S_o = channel slope. Hayami's diffusivity is properly a *kinematic hydraulic diffusivity*, ostensibly because it does not include inertia.

Hayami's relation, applicable to any channel, is:

$\nu_h = q_o / (2S_o)$

2. **Dooge (1973)** improved the concept by including inertia in the formulation, effectively a *dynamic* hydraulic diffusivity. His relation, applicable only to a hydraulically wide channel with Chezy friction, is:

   $\nu_h = [q_o / (2S_o)] \; [1 - (\mathbf{F}_o^2/4)]$

3. **Dooge and others (1982)** made the inertial component of the hydraulic diffusivity a function of β, the exponent of the rating. Their relation, applicable to a channel of any friction and cross-sectional shape, is:

   $\nu_h = [q_o / (2S_o)] \; [1 - (\beta - 1)^2 \, \mathbf{F}_o^2]$

4. **Ponce (1991a, 1991b)** improved the formulation of hydraulic diffusivity by making the inertial component a function of the Vedernikov number. His relation, applicable to a channel of any friction and cross-sectional shape, is:

   $\nu_h = [q_o / (2S_o)] \; [1 - \mathbf{V}_o^2]$

The question of how important the Vedernikov number is in the calculation of hydraulic diffusivity merits further discussion here. Hayami (1951) developed an approximate value for hydraulic diffusivity, without the inclusion of inertia. When inertia is important, Dooge's 1973 equation accounts for it, but limited to Chezy friction for a hydraulically wide channel. Dooge and others (1982) relaxed the latter requirement for channels of any frictional type (laminar, Manning, or Chezy) and cross-sectional shape. Ponce (1991) expressed the inertial component of the hydraulic diffusivity solely in terms of the Vedernikov number.

Ponce's (1991) formulation is useful when the wave under consideration is truly a mixed a kinematic-dynamic wave, a situation which turns out to be very uncommon in practice (**Ponce, 2023b**). However, its theoretical appeal, without unduly complicating the calculations, remains enticing. On that basis, Ponce's (1991) formulation is recommended for general use in flood routing applications.

6. STABLE CHANNEL DESIGN

The concept of Vedernikov number is extremely useful in channel design to assure hydrodynamic stability. The hydraulic design of flow in steep lined canals requires an evaluation of the Vedernikov number associated with the design discharge. If the calculated Vedernikov number exceeds unity, the possibility exists for the formation of "roll" waves (Fig. 2). These waves have also been referred to in the literature as "pulsating" waves, to denote that fact that they invariably occur as a "train of mass waves" traveling downstream at great, often dangerous, speeds (**Lighthill and Whitham, 1955**; **Ponce and Choque Guzman, 2019**).

Fig. 2 Roll waves in a steep lateral canal, Cabana-Mañazo irrigation, Puno, Peru.

The design objective should be to constrain the roll waves within the established channel boundaries at the adopted design discharge or, better yet, **to design the channel cross-section to avoid rolls waves altogether.** This requires a thorough understanding of the nature and behavior of roll waves. Of necessity, the analysis is based on the evaluation of the rating exponent β (Table 1). To avoid roll waves (Fig. 3), the design β should be the lowest possible value, conmensurate with other design criteria such as cost, project fooprint, and other ancillary considerations.

Courtesy of Jorge Molina Carpio

Fig. 3 Roll waves in the channelized Huayñajahuira river, La Paz, Bolivia (2016).

The feasible range of variation of β is 1.0 to 1.67 when using Manning friction, and 1.0 to 1.5 for Chezy friction. For a triangular cross-section, β = 1.33 for Manning friction and 1.25 for Chezy friction. Values of β close to, but exceeding 1.0 (for instance, β = 1.04) are not likely to develop roll waves. The reason is that β conditions the kinematic wave celerity to be greater than the mean flow velocity: $c_k = \beta\ u > u$. For values of β greatly exceeding 1, for example, β = 1.6, the possibility for the occurrence of roll waves is envisioned. The parameter β is seen to be the **only** hydraulic parameter capable of predicting the onset of a roll wave event accurately and effectively.

Click -here- to watch a video of a roll wave event in the channelized Huayñajahuira river, in La Paz, Bolivia, on December 11, 2021.

7. ONLINE CALCULATION

We describe a calculation of β using **ONLINECHANNEL15B**, an online tool specifically designed to calculate the value of β for a prismatic channel of rectangular, triangular, or trapezoidal cross-section. Two examples are presented, explained in **Box C**. The objective is to show the variation of the Vedernikov number **V** with β, confirming once more their direct relation.

Box C. Online calculation of rating exponent β and Vedernikov number **V**.

1. Example 1: High β, rectangular cross-section.

   Discharge = 100 m^3/s; bottom width b = 6 m; flow depth y = 1.638 m; side slope z = 0; Manning's n = 0.025; bottom slope S = 0.06.

   This data set resembles the flow conditions for the channelized Huaynajahuira river in La Paz, Bolivia (Fig. 4).

   The results of ONLINECHANNEL15B are shown in Fig. 5.

2. Example 2: Lower β, trapezoidal cross-section.

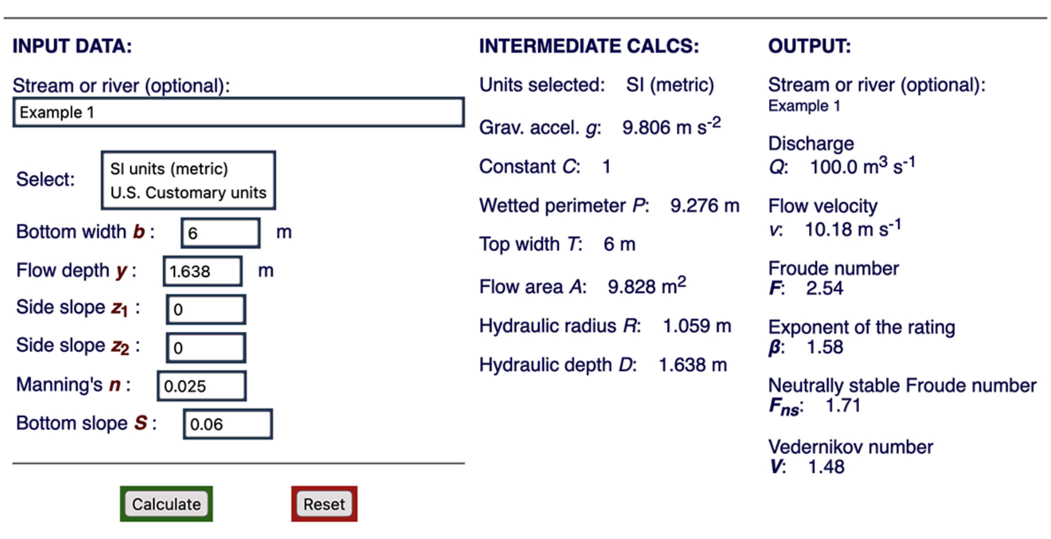
   Discharge = 100 m^3/s; bottom width b = 1 m; flow depth y = 3.503 m; side slope z = 0.5; Manning's n = 0.025; bottom slope S = 0.06.

   This hypothetical data set modifies Example 1 by reducing the bottom width and increasing the side slope.

   The results of ONLINECHANNEL15B are shown in Fig. 6.

Fig. 4 Channelized Huayñajahuira river, in La Paz, Bolivia.

[Click on top of figure to expand]



INPUT DATA:

Stream or river (optional): Example 1

Select: SI units (metric) / U.S. Customary units

Bottom width ***b*** : 6 m

Flow depth ***y*** : 1.638 m

Side slope z_1 : 0

Side slope z_2 : 0

Manning's ***n*** : 0.025

Bottom slope ***S*** : 0.06

Calculate   Reset

INTERMEDIATE CALCS:

Units selected: SI (metric)

Grav. accel. *g*: 9.806 m s^{-2}

Constant *C*: 1

Wetted perimeter *P*: 9.276 m

Top width *T*: 6 m

Flow area *A*: 9.828 m^2

Hydraulic radius *R*: 1.059 m

Hydraulic depth *D*: 1.638 m

OUTPUT:

Stream or river (optional): Example 1

Discharge
Q: 100.0 m^3 s^{-1}

Flow velocity
v: 10.18 m s^{-1}

Froude number
F: 2.54

Exponent of the rating
β: 1.58

Neutrally stable Froude number
F_{ns}: 1.71

Vedernikov number
V: 1.48

Fig. 5   Results of ONLINECHANNEL15B: Example 1.

[Click on top of figure to expand]

INPUT DATA:	INTERMEDIATE CALCS:	OUTPUT:
Stream or river (optional): Example 2	Units selected: SI (metric)	Stream or river (optional): Example 2
Select: SI units (metric) / U.S. Customary units	Grav. accel. g: 9.806 m s^{-2}	Discharge Q: 100.0 m^3 s^{-1}
Bottom width b: 1 m	Constant C: 1	Flow velocity v: 10.38 m s^{-1}
Flow depth y: 3.503 m	Wetted perimeter P: 8.832 m	Froude number **F**: 2.26
Side slope z_1: 0.5	Top width T: 4.503 m	Exponent of the rating β: 1.35
Side slope z_2: 0.5	Flow area A: 9.638 m^2	Neutrally stable Froude number F_{ns}: 2.77
Manning's n: 0.025	Hydraulic radius R: 1.091 m	Vedernikov number **V**: 0.81
Bottom slope S: 0.06	Hydraulic depth D: 2.140 m	
Calculate Reset		

Fig. 6 Results of ONLINECHANNEL15B: Example 2.

In Example 1, a rectangular channel, the results are: β = 1.58, and **V** = 1.48, indicating **unstable** flow. [It bears mentioning that the channelized Huayñajahuira river suffers from recurrent roll wave events, as documented by **Ponce and Choque Guzman (2019)**].

For Example 2, a trapezoidal channel, the results are: β = 1.35, and **V** = 0.81, indicating **stable** flow (**Ponce, 2021**). A direct relation is seen to exist between β and **V**; the lower the rating exponent β, the lower the Vedernikov number **V**.

8. CONCLUDING REMARKS

The Vedernikov number of open-channel hydraulics is reviewed, explained, and clarified. With the Froude number, they constitute an inseparable pair, more so now that their relation with β, the exponent of the discharge-area rating (**V**/**F** = β - 1) has been clearly identified and its usefulness in channel design amply demonstrated, A calculator for β and **V** in terms of hydraulic variables substantially increases the utility of the theory, making possible the avoidance of roll waves *at the design stage*.

APPENDIX

V. V. Vedernikov: Short Bio.

REFERENCES

Chow, V. T. 1959. Open-channel hydraulics. McGraw-Hill, Inc, New York, NY. **Extract: Chapter 8, Section 8**.

Lagrange, J. L. de. 1788. Mécanique analytique, Paris, part 2, section II, article 2, p 192.

Cornish, V. 1907. **Progressive waves in rivers**. *Journal of the Royal Geographical Society*, Vol. 29, No. 1, January, 23-31.

Craya, A. 1945. Calcul graphique des régimes variables dans lex canaux. *La Hoiulle Blanche*, No. 1, 39-60.

Craya, A. 1952. **The criterion for the possibility of roll wave formation**. *Gravity Waves, National Bureau of Standards Circular No. 521,* National Bureau of Standards, Washington, D.C. 141-151.

Dooge, J. C. I. 1973. **Chapter 9, Mathematical simulation of surface flow**, *Linear Theory of Hydrologic Systems*, Technical Bulletin No. 1468, Agricultural Research Service, U.S. Department of Agriculture, Washington, DC, October. Extract.

Dooge, J. C. I., W. B. Strupczewski, and J. J. Napiorkowski. 1982. **Hydrodynamic derivation of storage parameters of the Muskingum model**, *Journal of Hydrology,* Vol. 54, 371-387.

Hayami, I. 1951. **On the propagation of flood waves**. *Bulletin, Disaster Prevention Research Institute*, No. 1, December. Extract.

Lighthill, M. J. and G. B. Whitham. 1955. **On kinematic waves. I. Flood movement in long rivers**. *Proceedings, Royal Society of London, Series A,* 229, 281-316.

Nuccitelli, N. and V. M. Ponce. 2014. **The dynamic hydraulic diffusivity reexamined**. Online article.

Ponce, V. M. and D. B. Simons. 1977. **Shallow wave propagation in open channel flow**. *Journal of Hydraulic Engineering ASCE,* 103(12), 1461-1476.

Ponce, V. M. 1979. **On the classification of open channel flow regimes**. *Proceedings, Fourth National Hydrotechnical Conference*, Vancouver, British Columbia, Canada.

Ponce, V. M. 1991a. **New perspective on the Vedernikov number**. *Water Resources Research*, Vol. 27, No. 7, 1777-1779, July.

Ponce, V. M. 1991b. **The kinematic wave controversy**. *Journal of Hydraulic Engineering ASCE,* 117(4), 511-525.

Ponce, V. M. and P. J. Porras. 1995. **Effect of cross-sectional shape on free-surface instability**. *Journal of Hydraulic Engineering ASCE,* 121(4), 376-380.

Ponce, V. M. 2014. **Fundamentals of Open-channel Hydraulics**. Online text.

Ponce, V. M. and B. Choque Guzman. 2019. **The control of roll waves in channelized rivers**. Online article.

Ponce, V. M. 2021. **Design of a stable channel on a steep slope using the exponent of the rating**. Online article.

Ponce, V. M. 2023a. **The states of flow**. Online article. *https://ponce.sdsu.edu/the_states_of_flow.html*

Ponce, V. M. 2023b. **Kinematic and dynamic waves: The definitive statement**. Online article. *https://ponce.sdsu.edu/kinematic_and_dynamic_waves.html*

Powell, R. W. 1948. **Vedernikov's criterion for ultra-rapid flow**. *Transactions, American Geophysical Union,* Vol. 29, No. 6, 882-886.

Seddon, J. A. 1900. **River hydraulics**. *Transactions, ASCE,* Vol. XLIII, 179-243, June.

Vedernikov, V. V. 1945. **Conditions at the front of a translation wave disturbing a steady motion of a real fluid**. *Doklady Akademii Nauk USSR,* 48(4), 239-242.

Vedernikov, V. V. 1946. **Characteristic features of a liquid flow in an open channel**. *Doklady Akademmi Nauk USSR,* 52(3), 207-210.

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