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VEDERNIKOV'S CRITERION FOR ULTRA-RAPID FLOW

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Abstract--A criterion is given for the stability of steady uniform flow in open channels. When this number exceeds one, the flow is ultra-rapid, roll waves form, and the flow cannot be steady. This criterion is compared with several which have been proposed, and is judged to be more comprehensive and exact.

An empirical formula for Chezy's "C" in ultra-rapid flow is derived from data already published by the author.

Introduction--In two very interesting papers published in Russian but available in English, VEDERNIKOV [1945 and 1946] has developed a criterion for the stability of steady uniform flow. He says that when the criterion is less than one, waves tend to dampen out, but that when it is equal to or more than one, they amplify so that steady flow is impossible and we have what he calls ultra-rapid flow. Others have used the expressions roll waves or slug flow.

This criterion, which I am calling the Vedernikov number, may be written as

$$\underline{V} = (1 + \beta)MV/p(u - V) \qquad (1)$$

where V is the mean velocity across any section, u is the velocity of the wave, and β and p are exponents in the resistance equation

$$S = k\,V^p/R^{(1 + B)} \qquad (2)$$

and M measures the effect of the shape of the cross section and is defined by

$$M = 1 - R\,dP/dA \qquad (3)$$

where S = slope of energy gradient, R = hydraulic radius, P = wetted perimenter, and A = area of cross section. We also have the well known relationship

$$u - V = \sqrt{gA/\alpha B} \qquad (4)$$

where B = top width, and α is the coefficient which corrects for nonuniformity in the velocity distribution.

Development--In rough channels it is now known that for tranquil flow Chezy's C is constant for any one relative roughness, and in (2), p = 2 and $\beta = 0$. In smooth channel flow on the other hand, C is a function of the Reynolds number, which requires that $p + \beta = 2$, as can be seen by writing $S = k\,V^p/R^{(1+\beta)} = V^2/C^2R$ and $(C^2k = V^{(2 - b)}R^{\beta}$. Since in Reynolds number $\underline{R} = 4VR/\nu$, the velocity and the hydraulic radius have the same exponent, this requires that $2 - p = \beta$.

In his papers, Vedernikov has much of the time taken p = 2, without at the same time taking $\beta = 0$. In fact Manning's formula makes p = 2 and $\beta = 1/3$, but it is beginning to be realized that Manning's formula is a not-too-successful attempt to make the same law cover both smooth and rough channel flow. In what follows we will take $p + \beta = 2$, which leads to another form of the criterion

$$\underline{V} = (1 + \beta)MV/(2 - \beta)\sqrt{gA/\alpha B} \qquad (5)$$

If we define the Froude number $\underline{F}$ as the ratio of the mean velocity of flow to the velocity of a gravity wave in still water, we have from (4)

$$\underline{F} = V/\sqrt{gA/\alpha B} \qquad (6)$$

and substituting in (5) we have

$$V = (1 + \beta)MF/(2 - \beta) \quad \ldots\ldots (7)$$

and the critical value of F for which $V = 1$ is

$$F_c = (2 - \beta)/(1 + \beta)M \quad \ldots\ldots (8)$$

Combining (7) and (8), we have

$$V = F/F_c \quad \ldots\ldots (9)$$

For a rectangular channel, $A = By$, $P = B + 2y$, and $R = By/(B + 2y)$. Also $dP/dA = (dP/dv)/(dA/dy) = 2/B$, so that from (3), $M = 1 - 2y/(B + 2y) = B/(B + 2y) = 1/(1 + 2y/B)$. Therefore, from (8), for a rectangular channel

$$F_c = [(2 - \beta)/(1 + \beta)]\,(1 + 2y/B) \quad \ldots\ldots (10)$$

Comparison with other criteria--In a very narrow channel, F_c would be very large and ultra-rapid flow would hardly ever occur. On the other hand, in a very wide channel the $(1 + 2y/B)$ in (10) approaches one, and $F_c = (2 - \beta)/(1 + \beta)$. For rough channel flow, $\beta = 0$ and $F_c = 2$. This is the value given by JEFFREYS [1925] and by THOMAS [1939].

For smooth channel flow with $\beta = 0.20$, the wide channel gives a critical Froude number of $1.8/1.2 = 1.5$, which is the result obtained by KEULEGAN and PATTERSON [1940] by quite different reasoning.

Lea's formula for smooth channels gives $\beta = 0.25$, and $F_c = 1.40$ as given by ROBERTSON and ROUSE [1941].

The advantage of Vedernikov's criterion is that it is much more general than those heretofore proposed, applying to channels of any shape, and to both smooth and rough channel flow, and that it shows how to correct for the effect of non-uniform velocity distribution.

One interesting result is that for laminar flow in very wide channels (sheet flow), $\beta = 1$, $M = 1$, and $F_c = 0.50$. This seems to indicate that with increasing velocity, tranquil flow will change to ultra-rapid directly, without passing through the rapid stage. This result seems to be implied in the article by ROBERTSON and ROUSE [1941].

The question as to what value of β should be used in any given case may be answered as follows. For non-turbulent flow, $\beta = 1$ and $V = 2MF$. For turbulent rough-channel flow, $\beta = 0$ and $V = MF/2$. For turbulent smooth-channel flow, (2) may be combined with Chezy's formula to give $C^2 = (VR)^{\beta}/k = R^{\beta}\gamma^{\beta}/4^{\beta}k$. Differentiating this and making various substitutions we have

$$\beta = (2R/C)(dC/dR) \quad \ldots\ldots (11)$$

An as yet unpublished study by the writer indicates that for rapid flow in smooth channels

$$C = 41.2 \log_{10}(R/C) + 17.9\,\beta_k \quad \ldots\ldots (12)$$

where β_k is the shape correction used by KEULEGAN [1938, Eq. (53)...].

Regarding β_k as a constant and differentiating (12) gives $(dC/dR)[1 + 41.2 \log_{10}(e)/C] = 41.2 \log_{10} e/R$, which combined with (11) gives

$$\beta = 1/(0.0279\,C + 0.5) \quad \ldots\ldots (13)$$

Values from this formula are plotted in Figure 1. The values of R there given are from (12) with two different values of β_k. The upper line is for $\beta_k = 0$, which is the value for infinitely wide channels. The lower line is for $\beta_k = 0.19315$ and $17.9\,\beta_k = 3.46$, which is the maximum value for rectangular channels and occurs when the depth is half the width.

Experimental test--As a test of this criterion, some data obtained by the writer [POWELL, 1946] on flow in a rectangular flume were used. It should perhaps be noted that while by title the study dealt with rough channels, data were at the same time obtained on smooth channels. It was

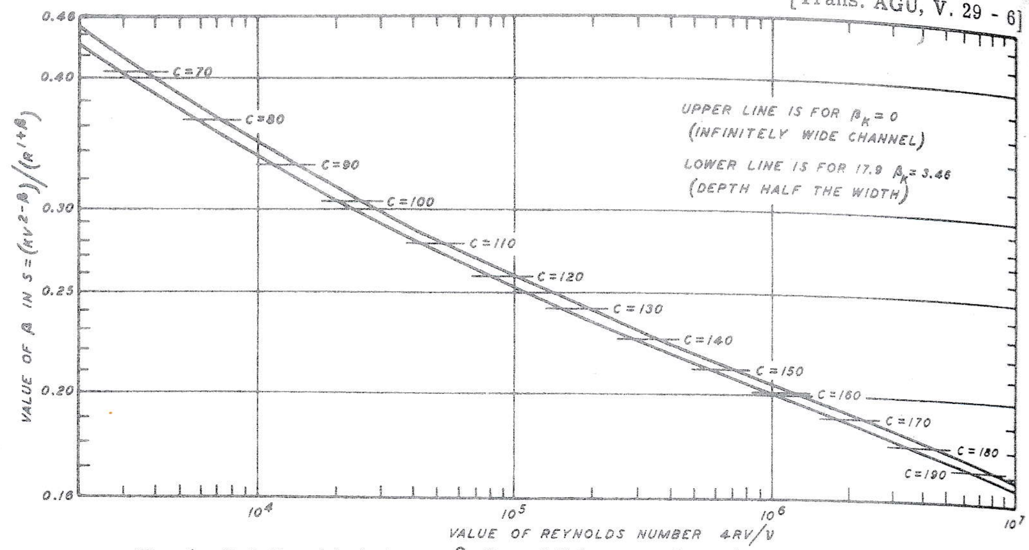

Fig. 1--Relationship between β, C, and $\underline{R}$ for smooth rectangular channels

found that all the runs in the smooth channel with the steepest slope (those in part (b) of Table 1 of POWELL [1946], gave $\underline{V}$ more than one, while all runs in the smooth channel with flatter slopes (those in part (a), and all those in the rough channels (parts (c) to (n) inclusive) gave $\underline{V}$ less than one, thus checking Vedernikov's theory. Incidentally the writer was mistaken in his classification in his earlier paper. The heading of part (a) should have been Tranquil and Shooting (or Rapid) Flow, and the heading of part (b), Ultra-rapid Flow. Runs 46 and 24 were tranquil, and Run 23, shooting, but both followed the same law of resistance. Runs 1, 2, 3, and 8 at the steepest slope were ultra-rapid with waves forming, and the law of resistance changed. Many of the runs in rough channels at the steepest slope, and part of the runs quoted from Bazin in part (n) of the table were also shooting flow with $\underline{F}$ more than 1.00, but for all of these $\underline{V}$ was less than 1.00. A few of these with the highest values are included in Table 1 of this article.

The method of computing Table 1 was as follows: For the rough channels, β was taken as zero, and for the smooth channels it was computed from the observed Chezy coefficient by (13). Using these values of β, the critical Froude number, $\underline{F}_c$, was computed by (10), the channel width B being 0.682 ft.and the values of the depth y being taken from Table 1 of POWELL [1946]. The Froude number $\underline{F}$ was computed from (6), which for rectangular channels reduces to $\underline{F} = V\sqrt{\alpha/gy}$. Values of V and y were taken from Table 1 of POWELL [1946], and g was taken as 32.16 ft/sec^2. Following KEULEGAN [1942], α was defined as the ratio of the mean of the squares of the velocities to the square of the mean velocity. Its actual value was unknown and probably varied from runto run, but for want of a better value it was taken as 1.02. The resulting values of $\underline{F}$ are this about one per cent larger than those given by POWELL [1946] where α was omitted from the definition of $\underline{F}$. The Vedernikov number, $\underline{V}$, was then computed by (9). To avoid computational errors from accumulating, the computations were made on a machine to four figures, although it is realized that the data are not nearly that accurate.

Formula for ultra-rapid flow--It seemed natural to suppose that the law of resistance for ultra-rapid flow would include the Vedernikov number. A statistical study of the data checked this assumption, but showed that the best fit required a term for $\underline{F}$ as well as one for $\underline{V}$. The resulting equation was

$$C = 41.2 \log_{10} (\underline{R}/C) + 42.3 \underline{F} - 21.8 \underline{V} - 113.7 \quad \ldots\ldots (14)$$

which by (9) can be put in the form

$$C = 41.2 \log_{10} (\underline{R}/C) + 42.3 \underline{V} (\underline{F}_c - 0.515) - 113.7 \quad \ldots\ldots (15)$$

It was not thought necessary to retain any c_s term, since the M in the Vedernikov number is supposed to evaluate the effect of shape. By means of a table of values of C + 41.2 $\log_{10}$ C, the values of C by (14) were computed and tabulated in Table 1, and the discrepancies from the observed values listed. The mean discrepancy is 2.15, instead of 2.44 as given by the former formula. The

Table 1--Data for

Run	Observed C
1- 1	103.4
1- 2	111.3
1- 3	115.6
1- 4	118.7
1- 5	123.5
2- 1	122.1
2- 2	116.1
2- 3	115.6
2- 4	118.3
2- 5	120.1
2- 6	114.5
2- 7	129.1
2- 8	127.4
2- 9	127.1
2-10	132.9
2-11	116.3
2-12	116.2
2-13	116.2
3- 1	110.3
3- 2	112.4
3- 3	111.6
3- 4	116.7
3- 5	118.9
3- 6	118.9
3- 7	127.2
3- 8	126.8
3- 9	126.9
8- 1	102.9
8- 2	106.7
8- 3	108.0
8- 4	112.1
8- 5	113.8
8- 6	114.3
8- 7	117.2
8- 8	120.5
8- 9	121.6
8-10	115.2
8-11	110.5
8-12	111.5
8-13	122.6
8-14	125.8
8-15	122.8
8-16	123.8
8-17	125.6
23- 1	125.8
23- 2	130.6
23- 3	137.7
23- 4	137.0
23- 5	143.1
23- 6	145.0
14- 1	73.7
14- 2	80.7
14- 3	79.3
14- 4	78.9
14- 5	80.4

Table 1--Data for computing Vedernikov Number, and computed value of Chezy's "C" by (14)

Run	Observed C	β	$\underline{F}_c$	F = 0.1781V/ y	$\underline{V} = \underline{F}/\underline{F}_c$	Computed C	Discrepancy −	Discrepancy +
1- 1	103.4	0.295	1.757	2.725	1.551	100.9	2.5	. .
1- 2	111.3	0.277	1.903	2.805	1.474	110.0	1.3	. .
1- 3	115.6	0.268	1.963	2.842	1.448	113.3	2.3	. .
1- 4	118.7	0.262	2.282	2.643	1.158	117.7	1.0	. .
1- 5	123.5	0.253	2.445	2.634	1.077	121.1	2.4	. .
2- 1	122.1	0.256	1.899	3.311	1.744	124.5	. .	2.4
2- 2	116.1	0.267	1.841	3.081	1.674	115.0	1.1	. .
2- 3	115.6	0.268	1.830	3.040	1.661	113.3	2.3	. .
2- 4	118.3	0.263	1.823	3.108	1.705	114.5	3.8	. .
2- 5	120.1	0.260	1.863	3.154	1.693	117.9	2.2	. .
2- 6	114.5	0.271	1.818	3.021	1.662	112.4	2.1	. .
2- 7	129.1	0.244	2.228	3.060	1.373	129.5	. .	0.4
2- 8	127.4	0.247	2.227	3.044	1.367	129.2	. .	1.8
2- 9	127.1	0.247	2.231	3.047	1.366	129.6	. .	2.5
2-10	132.9	0.238	2.554	2.818	1.103	130.0	2.9	. .
2-11	116.3	0.267	1.889	2.974	1.574	115.0	1.3	. .
2-12	116.2	0.267	1.941	3.038	1.565	119.6	. .	3.4
2-13	116.2	0.267	1.933	3.021	1.563	118.6	. .	2.4
3- 1	110.3	0.280	1.738	2.942	1.693	106.5	3.8	. .
3- 2	112.4	0.275	1.868	2.935	1.571	113.5	. .	1.1
3- 3	111.6	0.277	1.891	2.946	1.558	115.2	. .	3.6
3- 4	116.7	0.266	1.996	2.885	1.445	117.2	. .	0.5
3- 5	118.9	0.262	2.104	2.891	1.374	121.3	. .	2.4
3- 6	118.9	0.262	2.165	2.914	1.346	124.5	. .	5.6
3- 7	127.2	0.247	2.420	2.711	1.120	124.1	3.1	. .
3- 8	126.8	0.248	2.463	2.673	1.085	123.9	2.9	. . .
3- 9	126.9	0.247	2.556	2.705	1.058	127.1	. .	0.2
8- 1	102.9	0.297	1.752	2.755	1.572	103.9	. .	1.0
8- 2	106.7	0.288	1.827	2.766	1.514	107.7	. .	1.0
8- 3	108.0	0.285	1.880	2.769	1.473	110.1	. .	2.1
8- 4	112.1	0.276	1.954	2.777	1.421	113.0	. .	0.9
8- 5	113.8	0.272	2.011	2.766	1.375	114.8	. .	1.0
8- 6	114.3	0.271	2.058	2.745	1.334	115.9	. .	1.6
8- 7	117.2	0.265	2.113	2.739	1.296	117.2	0.0	0.0
8- 8	120.5	0.259	2.166	2.750	1.270	119.0	1.5	. .
8- 9	121.6	0.257	2.216	2.739	1.236	120.4	1.2	. .
8-10	115.2	0.269	2.244	2.627	1.171	118.0	. .	2.8
8-11	110.5	0.279	2.265	2.541	1.122	116.2	. .	5.7
8-12	111.5	0.277	2.307	2.520	1.092	116.6	. .	5.1
8-13	122.6	0.255	2.381	2.623	1.102	121.0	1.6	. .
8-14	125.8	0.249	2.417	2.641	1.093	122.1	3.7	. .
8-15	122.8	0.255	2.446	2.565	1.049	120.4	2.4	. .
8-16	123.8	0.253	2.506	2.607	1.040	123.3	0.5	. .
8-17	125.6	0.250	2.558	2.616	1.023	124.6	1.0	. .
							46.9	47.5
23- 1	125.8	0.249	1.887	1.711	0.907	. . . .	. . .	. . .
23- 2	130.6	0.241	2.040	1.692	0.829	. . . .	. . .	. . .
23- 3	137.7	0.230	2.342	1.625	0.694	. . . .	. . .	. . .
23- 4	137.0	0.231	2.516	1.584	0.630	. . . .	. . .	. . .
23- 5	143.1	0.223	2.804	1.553	0.554	. . . .	. . .	. . .
23- 6	145.0	0.220	3.158	1.464	0.464	. . . .	. . .	. . .
14- 1	73.7	0	2.704	1.970	0.729	. . . .	. . .	. . .
14- 2	80.7	0	2.832	2.123	0.749	. . . .	. . .	. . .
14- 3	79.3	0	3.138	1.986	0.634	. . . .	. . .	. . .
14- 4	78.9	0	3.402	1.891	0.556	. . . .	. . .	. . .
14- 5	80.4	0	3.760	1.820	0.484	. . . .	. . .	. . .

improvement is hardly enough to be significant, but it may be noted that in 34 of the 44 runs, the fit is improved. Equations (14) and (15) must be regarded as tentative until a larger number of more accurate measurements are available. It is entirely possible that an equation of different form would fit the data better. But it seems quite certain that the resistance law is different in ultra-rapid flow than in rapid, and that in the former, C depends upon the Vedernikov number.

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